# *SPINOFFS*

Spinoffs are relatively short learning modules inspired by the LTAs. They can be easily implemented to support student learning in courses ranging from prealgebra through calculus. The Spinoffs typically give students an opportunity to use mathematics in a real world context.

LTA - SPINOFF 16A — Modeling The Space Shuttle Landing: The Cubic Spline

LTA - SPINOFF 16B — Modeling The Space Shuttle Landing: The Circular Pull-Up

LTA - SPINOFF 16C — Knots and Machs

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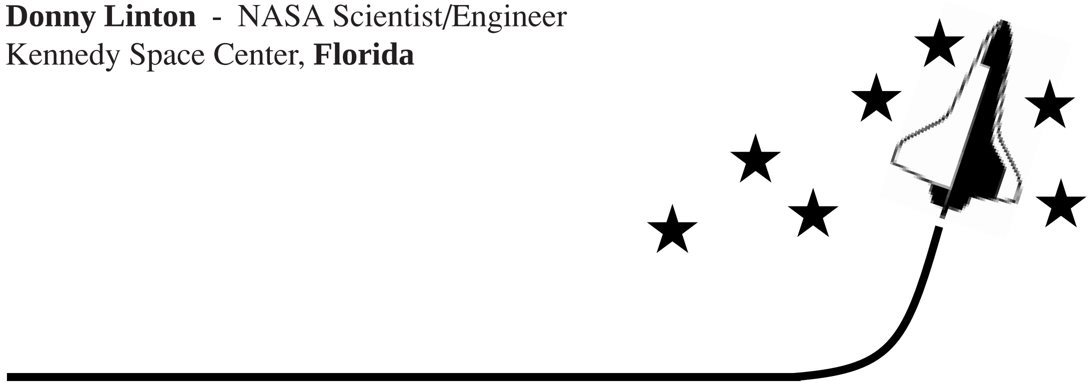

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## SPINOFF 16B

## Modeling The Space Shuttle Landing: The Circular Pull-Up

Space Shuttle missions have become a familiar topic on the evening news and undoubtedly the fiery launch of a Shuttle from the Kennedy Space Center is a marvel to behold. However, the final and most critical phases of a space shuttle mission are deorbit, re-entry into the atmosphere, and landing. This Spinoff will focus on the final landing approach of the Shuttle, but first here is a brief description of the events from deorbit until final approach.

The Space Shuttle has no power and essentially acts as a glider, although it is much heavier and has a less aerodynamic shape than an actual glider (it has been referred to as a "flying brick"). It has only one chance to land since, without engines, it cannot climb and try another approach. The process begins half a world away from the landing runway, when the Space Shuttle is traveling 200 miles above the ground at a speed of over 17,000 miles per hour. It is now about 60 minutes to touchdown. During the deorbit burn, the Space Shuttle travels tail first and loses some speed and altitude. Once the burn is complete, the Space Shuttle is reversed, its nose is raised, and the atmospheric entry begins. It is now about 31 minutes to touchdown. During this phase, there is a tremendous heat buildup around the Shuttle and portions of the vehicle's exterior reach 2,800°F (you have probably heard about the tiles used on the surface of the vehicle to protect it at this critical time). The heat strips electrons from the air around the Space Shuttle, enveloping it in a sheath of ionized air that blocks all communication with the ground for about 12 minutes. During this interval the pilot will perform several banking maneuvers called roll reversals or S-turns to control descent. When the Space Shuttle comes out of the communications blackout, its speed is about 8,275 mph and 12 minutes remain to touchdown. It is now committed to a particular landing site and must begin the final approach with enough altitude and speed to reach the touchdown point. At this point the vehicle travels a circular path around an imaginary cone that will line it up with the center line of the runway. Once the Shuttle comes out of this turn, it is ready for its final approach to the runway.

### The Final Approach to the Runway

In this Spinoff we will examine the path the Shuttle takes on this final approach to the runway. Coming out of its turn, the Shuttle should be at an altitude of 13,365 ft, have a speed of 424 mph, and be 7.5 miles (horizontal distance) from the runway. It is now 86 sec to touchdown. The nose is down so that the space shuttle can descend steeply to a point 7,500 ft from the runway threshold, when its altitude should be 1,750 ft. The vehicle then enters a transitional phase. The Shuttle's nose is raised as it heads for a position where its altitude is 131 ft and its distance from the runway threshold is 2,650 ft. The Shuttle is now 17 sec to touchdown. From here the Space Shuttle enters the final phase, aiming at a point 2,200 ft down the runway.

In LTA 16, the flight path satisfying these conditions was modeled by a function with an exponential piece and two linear pieces as shown in Figure 1 on the next page.

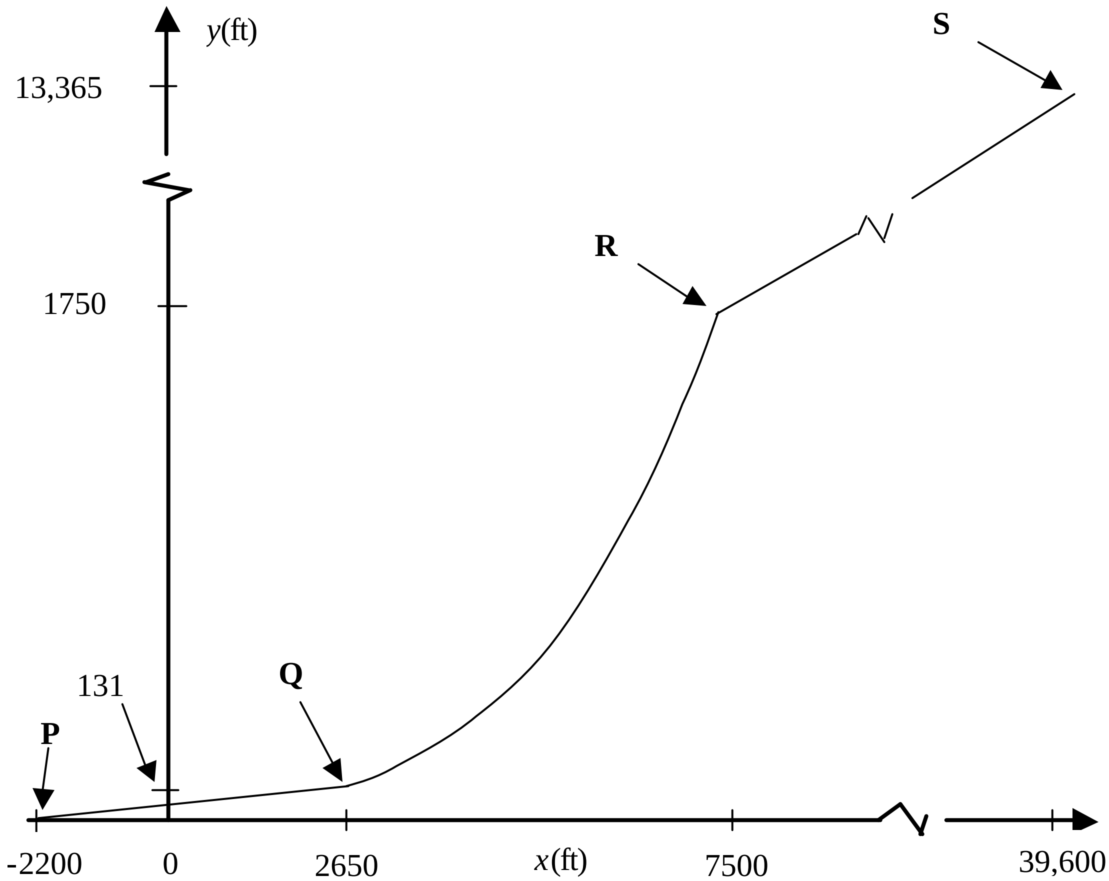


**Figure 1 Shuttle Flight Path - Approach to Runway**

In Figure 1, the Shuttle approaches the runway from right to left. RS is phase 1, QR is the transitional phase, and PQ is the final phase. The runway threshold is at the origin and the touchdown point is at P.

The following piecewise function, $F$, describes the flight path given by the graph in Figure 1.

$$y = F(x) = \begin{cases} 0.3618380062x - 963.7850467 & \text{if } 7{,}500 \le x \le 39{,}600 \\ 31.78056592(1.000534612)^{x} & \text{if } 2{,}650 \le x < 7{,}500 \\ 0.02701x + 59.42268 & \text{if } -2{,}200 \le x < 2{,}650 \end{cases}$$

Notice that the function, $F$, is composed of two line segments connected by an exponential curve.

**Exercises**

1) Examine and discuss the graph of $F$ shown in Figure 1. Can you suggest any reason why the graph could not be a completely accurate model of the Space Shuttle's approach to the runway? Hint: Look at the left-hand and right-hand derivatives at 2,650 ft and 7,500 ft.

In Exercise 1 you probably concluded that the points Q and R are problematic. At each of these points, the "right slope" is not equal to the "left slope". That is, the function does not have a derivative at Q and R. The Shuttle would have to abruptly change direction at points Q and R. This cannot happen in the real world. Since the situations at points Q and R are similar, this Spinoff will address the solution for only point R. Your task in this LTA is to replace a section of the curve that contains Point R by a curve that smooths out the sharp point. That is, you will modify the given function in a neighborhood of R to produce a new function that has a derivative at point R.

2) Brainstorm some ways of changing the graph near R so that it would represent a reasonable flight path.

Since the exponential piece does not produce a function that is differentiable at the two endpoints of the transitional phase, we will try a different shape. An exponential piece was originally used because a NASA training pilot had referred to this phase as the "exponential capture". However, another training pilot used the phrase "circular pull-up" for this same transitional phase. In this Spinoff, we will work with a circular arc to smooth out the transition at point, R, between the first linear phase and the exponential curve. To do this, we need to find the equation of a circle that is tangent to the exponential piece at some point near R and is also tangent to the first phase linear piece near R. We will begin by choosing a point, U, on the exponential piece where $x = 7400$. The circle must be tangent to the exponential piece at U. We will then use this point to find the point, W, where the first linear piece will also be tangent to the circle. Once this is done we will need to find the center of the circle and the radius. (The same method could be used to smooth out the curve at point Q.)

Depending on the starting point, U, (the left end of the circular arc), there are many solutions for this problem. The following solution patches in a circular arc that is tangent to the exponential phase of the graph at the point, U, whose horizontal distance is 7,400 ft from the threshold of the runway. The arc will also be tangent to the first linear piece of the graph at a point that we will denote by W. In Exercises 3-12, you will find the equation of a circular arc that is tangent to the exponential part of the graph at U(7400, 1658.923514) and tangent to the first linear phase at W.

**Note:** As you perform the calculations, do <u>not</u> round off answers that will be used in a subsequent calculation. That is, at each step use the full decimal place accuracy provided by your graphing calculator. You can do this by storing all intermediate answers in variables on the calculator.

## Exercises

You should make and label a geometric diagram to accompany your step-by-step algebraic solution. Start by sketching the first and second phases of the flight path and then add to it as you do the following exercises. **Note:** To simplify the following calculations, it is advisable to use a computer algebra system such as Derive™ or the algebra system on the Texas Instruments TI-89™ and TI-92™ graphing calculators,. If you use a computer algebra system, set the display digits to 12.

3) Find the equation of the line, $L_1$, that is tangent to the graph of *F* at the point U(7400, 1658.923514).
(Hint: You will need to calculate the derivative of the exponential part of *F*.)
Equation of $L_1$: ______________________________________

4) Let $L_2$ represent phase 1 of the graph *F*. Lines $L_1$ and $L_2$ will be tangents of the circle that you construct. As a first step, determine the coordinates of the point of intersection, V, of $L_1$ and $L_2$.
V ( ________________ , ________________ )

5) Find the distance, d, between U and V.
d = __________________

6) You will now find a point W on line L2 so that the length of line segment WV is equal to the length of line segment UV. Find the coordinates of the point W on $L_2$ whose distance from V is equal to d. (Hint: Find the equation of the circle with center V and radius d. Solve the equation of this circle simultaneously with the equation for $L_2$. You will need to solve a quadratic equation. Be sure to choose the sign in the quadratic formula that yields a solution on the graph of the first phase.)
W( ________________ , ________________ )

In preparation for the Exercises 7-12, construct a line $L_3$ perpendicular to line $L_2$ at point W. Then construct a line $L_4$ perpendicular to line $L_1$ at point U. Let C represent the point of intersection of lines $L_3$ and $L_4$. Connect points C and V with a line segment. Observe that the right triangles CUV and CWV are congruent. This implies that $L_3$ and $L_4$ are of equal length. As a result, you can now construct a circle with center C that is tangent to $L_1$ and $L_2$ at point U and W respectively.

7) Find the equation of the line $L_3$ that is perpendicular to the graph of *F* at the point W.
Equation of $L_3$: ____________________________________________

8) Find the equation of the line $L_4$ that is perpendicular to the graph of $F$ at the point U.
Equation of $L_4$: ______________________________

9) Find the coordinates of the point C where $L_3$ and $L_4$ intersect.
C( ____________ , ____________ )

10) Find the distance r between C and U.
r = ________________

11) Write the equation of the circle with center C and radius r.
Equation of Circle: ______________________________

12) Solve the equation of the circle you obtained in Exercise 11 for $y$ in terms of $x$.
(Hint: When you solve for $y$ you will obtain two solutions. You will need to choose the sign that gives the upper arc of the circle.)
$y$ = ______________________________________

13) Graph the flight path with the circular arc patched in near R.

The circular arc that you just found depended on the position of the starting point, U, on the exponential section of the graph. If you start with another point and use the same procedure, you will find a different arc to replace a portion of the curve that contains the sharp point R. The procedure you used to find the arc that is tangent to the curve at the point with first coordinate 7,400 ft can be used to write the following program for a TI-83™. The program below labeled “CORNERS” will find the circular arc for any given starting point U on the exponential section of the Shuttle path if U is not too far from the point R. The program will also find the coordinates of the points U and W where the arc is tangent to the graph of *F*.

```
PROGRAM:CORNERS
: Disp "ENTER AN
X VALUE BETWEEN
7000 AND 7450"
: INPUT Z
: (1750/131)^(1/4
850)?  A
: 131/(A^2650)?  B
: B*A^Z*ln(A)?  C
: B*A^Z?  D
: -(D – C*Z)?  E
: (13365 – 1750)/(3
9600 – 7500)?  F
: -(1750 – F*7500)?
G
: (E – G)/(C – F)?  H
: F*H – G?  I
: v((H – Z)^2 + (I –D)
^2)?  J
: G+I?  K
: 1+F^2?  L
: 2*H+2*F*K?  M
: H^2+K^2 –J^2?  N
: (M+v(M^2 –4*L*N)
)/(2*L)?  O
: F*O – G?  P
: 1/C?  Q
: Q*Z+D?  R
: 1/F?  S
: S*O+P?  T
: (T – R)/(S –Q)?  U
: -Q*U+R?  V
: v((O – U)^2 + (P – V)^2
)?  W
: ClrHome
```

```
: Disp"Y=A+v(B^2 –
(X –C)^2)"
:DisP "WHERE A,B
,C=",V,W,U
: Disp"********"
: Disp "PRESS ENT
ER"
: Pause
: ClrHome
: Disp"THE ARC I
NTER-"
: Disp"SECTS THE
 GRAPH"
: Disp"OF F IN T
HE"
: Disp"POINTS U
AND W."
: Disp"******"
: Disp "PRESS ENT
ER"
: Pause
: ClrHome
: Disp"Point U(X
,Y)=",Z,D
: :Disp"POINT W(X
,Y)=",O,P
: Stop
```

**Exercises**

14) Run Program CORNERS with X = 7,400. How does the output compare with your answer in Exercise 12?

15) Run Program CORNERS with X = 7,000 to find the circular arc function, *g*, that is tangent to the graph of *F* at the point with x-coordinate equal to 7,000. The program will also give you the coordinates of the points of intersection of the graphs of *F* with the circular arc function. Evaluate the derivatives of the circular arc function and the function, *F*, at the two values of x where they intersect. Are the values of the derivatives of *g* and *F* at the left endpoint of the arc equal? Are the derivatives of the two functions equal at the right endpoint of the arc? What do the values of the derivatives at the endpoints of the arc tell you about the flight path at those points?