SPINOFFS

Spinoffs are relatively short learning modules inspired by the LTAs. They can be easily implemented to support student learning in courses ranging from prealgebra through calculus. The Spinoffs typically give students an opportunity to use mathematics in a real world context.

LTA - SPINOFF 16A	Modeling The Space Shuttle Landing: The Cubic Spline
LTA - SPINOFF 16B	Modeling The Space Shuttle Landing: The Circular Pull-Up
LTA - SPINOFF 16C	Knots and Machs

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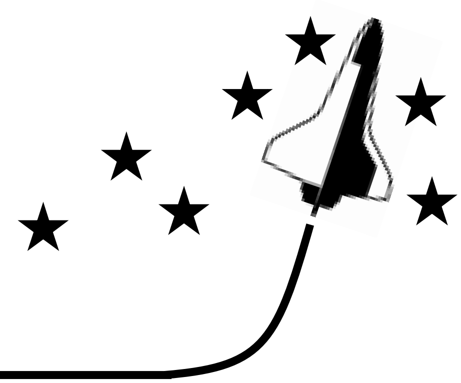

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SPINOFF 16A

Modeling The Space Shuttle Landing: The Cubic Spline

Space Shuttle missions have become a familiar topic on the evening news and undoubtedly the fiery launch of a Shuttle from the Kennedy Space Center is a marvel to behold. However, the final and most critical phases of a space shuttle mission are deorbit, re-entry into the atmosphere, and landing. This Spinoff will focus on the final landing approach of the Shuttle, but first here is a brief description of the events from deorbit until final approach.

The Space Shuttle has no power and essentially acts as a glider, although it is much heavier and has a less aerodynamic shape than an actual glider (it has been referred to as a "flying brick"). It has only one chance to land since, without engines, it cannot climb and try another approach. The process begins half a world away from the landing runway, when the Space Shuttle is traveling 200 miles above the ground at a speed of over 17,000 miles per hour. It is now about 60 minutes to touchdown. During the deorbit burn, the Space Shuttle travels tail first and loses some speed and altitude. Once the burn is complete, the Space Shuttle is reversed, its nose is raised, and the atmospheric entry begins. It is now about 31 minutes to touchdown. During this phase, there is a tremendous heat buildup around the Shuttle and portions of the vehicle's exterior reach 2,800°F (you have probably heard about the tiles used on the surface of the vehicle to protect it at this critical time). The heat strips electrons from the air around the Space Shuttle, enveloping it in a sheath of ionized air that blocks all communication with the ground for about 12 minutes. During this interval the pilot will perform several banking maneuvers called roll reversals or S-turns to control descent. When the Space Shuttle comes out of the communications blackout, its speed is about 8,275 mph and 12 minutes remain to touchdown. It is now committed to a particular landing site and must begin the final approach with enough altitude and speed to reach the touchdown point. At this point the vehicle travels a circular path around an imaginary cone that will line it up with the center line of the runway. Once the Shuttle comes out of this turn, it is ready for its final approach to the runway.

The Final Approach to the Runway

In this Spinoff we will examine the path the Shuttle takes on this final approach to the runway. Coming out of its turn, the Shuttle should be at an altitude of 13,365 ft, have a speed of 424 mph, and be 7.5 miles (horizontal distance) from the runway. It is now 86 sec to touchdown. The nose is down so that the space shuttle can descend steeply to a point 7,500 ft from the runway threshold, when its altitude should be 1,750 ft. The vehicle then enters a transitional phase. The Shuttle's nose is raised as it heads for a position where its altitude is 131 ft and its distance from the runway threshold is 2,650 ft. The Shuttle is now 17 sec to touchdown. From here the Space Shuttle enters the final phase, aiming at a point 2,200 ft down the runway.

Note: You will need to work with angles measured in degrees and calculate with decimal numbers using many digits. You will need access to a calculator or computer program that solves equations. Instructions will be given for a TI-89™, TI-92™, and *Derive*™. Before you begin, if you are using a TI-89™ or a TI-92™, press MODE, set the DISPLAY DIGITS to **12**, and set ANGLE to **degree**. If you are using *Derive*™, click on Declare, Algebra State, Simplification. Then change the Angular Unit to **degree**, and change the number of Precision Digits to **12**. Whatever technology you use, enter all available digits for your function and use all available digits throughout.

1) The four data points given in the paragraph describing the final approach are (39600, 13365), (7500, 1750), (2650, 131), and (−2200, 0). This assumes that the origin is the runway threshold. Notice that in this coordinate system, the Shuttle would be coming in from right to left.

   The flight path between the first and second points is linear and the equation is:
   $y = 0.36184x - 963.7805$

   The flight path between the last two data points is also linear and the equation is:
   $y = 0.02701x + 59.42268$.

   In between these two linear paths, the Shuttle is in a transitional phase in which the orientation changes from nose down to nose up. Since a NASA training pilot had referred to this phase as the "exponential capture" phase, it seemed reasonable to model this phase with an exponential function. Its equation is:
   $y = 31.78057(1.000534612)^x$

   These three pieces, put together as a piecewise function, provide a model for the final landing path of the Shuttle. This is the piecewise function:

$$f(x) = \begin{cases} 0.02701x + 59.42268 & \text{if } -2{,}200 \le x \le 2{,}650 \\ 31.78057(1.000534612)^x & \text{if } 2{,}650 < x < 7{,}500 \\ 0.36184x - 963.78505 & \text{if } 7{,}500 \le x \le 39{,}600 \end{cases}$$

   Author this function in your calculator or *Derive*™. Here are instructions:

   Derive™
   Author the function as follows:
   If (x > −2200 and x ≤ 2650, 0.02701x + 59.42268, if (x > 2650 and x ≤ 7500, 31.78057*1.000534612^x, if (x > 7500 and x < 39600, 0.36184x − 963.78505)))

TI-89™ or TI-92™

The easiest way to author this function is to put one piece on each of three function lines.
In y1(x), enter 0.02701x + 59.42268 | x > −2200 and x ≤ 2650.
In y2(x), enter 31.78057*1.000534612^x | x > 2650 and x ≤ 7500.
In y3(x), enter 0.36184x − 963.78505 | x > 7500 and x < 39600.

Once you have entered your equations, use the window: $-2250 \le x \le 40000$ and $-500 \le y \le 14000$ to graph.

2) Notice that the graph is dominated by the right-hand line segment. In order to get a better "view" of the rest of the graph, adjust the window to cut off some of this line segment. Change the window so that the ymax is **7000** and the xmax is **20000**. Print this graph out.

3) The slope of each line segment gives you some idea about the steepness of the Shuttle's descent during this phase, but NASA flight engineers prefer to use *glide slope*. Glide slope is the angle, measured in degrees, which the line makes with the horizontal. Use trigonometry to find the glide slope during each of the linear phases. What is the relationship between the glide slope and the slope of the line?

4) The exponential function in the transitional phase was supposed to produce a path with a steep glide slope on entry and a shallow glide slope on exit. We will examine the exponential function in the model to see what glide slopes it produces. For a non-linear graph, the glide slope will change since the slope of the graph is changing. The slope of the graph is, of course, the derivative function. Find the derivative of the exponential piece.

5) What is the relationship between the glide slope during the exponential piece and the derivative?

6) Use the derivative and your answer to Exercise 5 to find the glide slope when x = 7,500 (the entry into the transitional phase). How does this compare to the glide slope of the linear path right before entry? What implication does this have for the differentiability of your piecewise function at x = 7,500?

7) Now use the derivative of the exponential piece and your answer to Exercise 5 to find the glide slope when x = 2,650 (the end of the transitional phase). How does this compare to the glide slope of the linear path right after exit? What implication does this have for the differentiability of your piecewise function at x = 2,650?

As you can see, the exponential function does not provide a differentiable model. When we constructed this function, we only required that the function contain the two points (2650, 131) and (7500, 1750). In this Spinoff we will investigate a third degree polynomial and we will take into account not only the two points but the derivative values at those two points as well.

The Cubic Spline

Our goal now is to find a function that satisfies four conditions: the two points, (2650, 131) and (7500, 1750), must belong to the function and the function must also satisfy the two derivative (slope) conditions. Mathematicians who work with curve-fitting use a cubic polynomial when these four conditions are present and the cubic polynomial is called a *cubic spline.* Cubic splines are used in computer graphics programs and in industrial design. We will begin by writing our third degree polynomial:

$$y = f(x) = ax^3 + bx^2 + cx + d$$

8) Use this function and the entry and exit points to write two equations in a, b, c, and d.

9) Write the derivative of the cubic polynomial.

10) What should the value of the derivative be at x = 2,650 and x = 7,500?

11) Use the derivative and the two values to write two more equations in a, b, and c.

You now have a system of four equations in four unknowns. Although you could solve this system algebraically if you used great care, it would be much easier to use the algebraic capabilities of the TI-89™, TI-92™, or *Derive*™. In this case, we will use matrix methods to solve the system of equations. If you have not done this before, think about your experience doing polynomial long division and then learning synthetic division. The idea was to eliminate unnecessary writing and focus only on the steps of an algorithm that would give the final answer. Using matrix methods for solving a system of equations also eliminates writing unnecessary variables and uses a series of calculations to gain the solution. Let's begin with a very simple system of equations to see how the method works.

12) Solve the system $\begin{cases} 4x + 3y = 1 \\ 3x - y = -9 \end{cases}$ by hand. You should get (–2, 3) as your solution.

A matrix is simply an array of numbers in rows and columns. The matrix of the coefficients of this system would look like this:

$$\begin{bmatrix} 4 & 3 \\ 3 & -1 \end{bmatrix}$$

However, we will set up what is called an *augmented matrix* for this system of equations. The matrix we will write has the coefficients on the left and the constants on the right. For this system, this is the augmented matrix:

$$\begin{bmatrix} 4 & 3 & \vdots & 1 \\ 3 & -1 & \vdots & -9 \end{bmatrix}$$

The goal is to use valid operations on rows to transform this into a matrix that looks like this: $\begin{bmatrix} 1 & 0 & \vdots & p \\ 0 & 1 & \vdots & t \end{bmatrix}$. This is called the *reduced row echelon* form of the matrix. Once we do this, the solution will be $x = p$ and $y = t$. Valid row operations are multiplication of a whole row by a number, addition of one row to another, or any combination of row multiplication and addition to achieve the desired result.

For this matrix, first multiply row 1 by $\frac{1}{4}$ to get a 1 in the first row and column. This gives:

$$\begin{bmatrix} 1 & \frac{3}{4} & \vdots & \frac{1}{4} \\ 3 & -1 & \vdots & -9 \end{bmatrix}$$

Next, multiply row 1 by -3 and add it to row 2 to get a 0 in the second row first column. This gives:

$$\begin{bmatrix} 1 & \frac{3}{4} & \vdots & \frac{1}{4} \\ 0 & \frac{-13}{4} & \vdots & -\frac{39}{4} \end{bmatrix}$$

Next, multiply row 2 by $-\frac{4}{13}$ to get a 1 in the second row second column. This gives:

$$\begin{bmatrix} 1 & \frac{3}{4} & \vdots & \frac{1}{4} \\ 0 & 1 & \vdots & 3 \end{bmatrix}$$

Finally, multiply row 2 by $-\frac{3}{4}$ and add it to row 1 to get a 0 in the first row second column. This gives:

$$\begin{bmatrix} 1 & 0 & \vdots & -2 \\ 0 & 1 & \vdots & 3 \end{bmatrix}$$

We now have the reduced row echelon form of the matrix, so our solution is:

$$x = -2 \text{ and } y = 3.$$

With simple coefficients and a small matrix, the method can be done easily by hand. However, you can see that the computations can quickly get very messy. Computer algebra systems have a built-in function that will automatically find the reduced row echelon form of an augmented matrix. You only need to enter the matrix and execute the command.

Following are instructions for the TI-89™, the TI-92™, and *Derive*™. The example will be the simple one we worked with above.

The TI-89™ and Reduced Row Echelon Form

First, the matrix is entered using the data/matrix editor.
Press APPS then 6:DATA/MATRIX EDITOR, then 3: NEW.
Under TYPE, press the ▸ and choose 2:MATRIX. Then press ▾ to VARIABLE and use the ALPHA key to type test. This gives your matrix the name test. Press ▾ and enter **2** for the row dimension and then **3** for the column dimension. Then press ENTER.
You can now enter the numbers into their correct locations in the matrix.

You are now ready to get the reduced row echelon form of the matrix. The command is *rref*. Now that the matrix is entered, press the HOME key and on the authoring line use the ALPHA key to type the command **rref (test)**. You should now have the reduced row echelon form.

The TI-92™ and Reduced Row Echelon Form

First, the matrix is entered using the data/matrix editor.
Press APPS then 6:DATA/MATRIX EDITOR, then 3: NEW.
Under TYPE, press the ▸ and choose 2:MATRIX. Then press ▾ to VARIABLE and type test. This gives your matrix the name test. Press ▾ and enter **2** for the row dimension and then **3** for the column dimension. Then press ENTER. You can now enter the numbers into their correct locations in the matrix.

You are now ready to get the reduced row echelon form of the matrix. The command is *rref*. Now that the matrix is entered, press the HOME key and on the authoring type the command **rref (test)**. You should now have the reduced row echelon form.

***Derive*™ and Reduced Row Echelon Form**

First, the matrix must be authored.
Click on Author, then Matrix, then enter **2** for the number of rows and **3** for the number of columns.
Now enter the numbers into the matrix.
Next, click on Author, then Expression, and type **row_reduce** and then press the **F3** function key to bring up the matrix. Then click on Simplify.
You should now have the reduced row echelon form.

Now that you are familiar with the process, we can go on to solve the system of equations you wrote earlier.

13) Write down the augmented matrix generated by the system of equations. In your derivative equations, since there is no *d* variable, the value is 0.

14) Now use the method above to find the reduced row echelon form of this matrix. If you are using a TI-89™ or TI-92™, you can create a new matrix called **deg3**. After you have entered the matrix, execute the command **rref(deg3)**.

15) You now have the values of *a, b, c,* and *d*. Write the solution to the system of equations.

16) Write the third degree polynomial model for the transitional phase.

17) To check that this is correct, differentiate this polynomial and evaluate the derivative at $x = 2{,}650$ and $x = 7{,}500$. Do these derivatives match the slopes of the linear pieces on either side?

18) Edit the function you authored in the beginning of the lab and replace the exponential function with the polynomial function. Then graph the new piecewise function and print out this graph.