SPINOFFS

Spinoffs are relatively short learning modules inspired by the LTAs. They can be easily implemented to support student learning in courses ranging from prealgebra through calculus. The Spinoffs typically give students an opportunity to use mathematics in a real world context.

LTA - SPINOFF 11A — Newton's Law of Cooling and Heating: Vaporization of Liquid Helium

LTA - SPINOFF 11B — The Combined Gas Law

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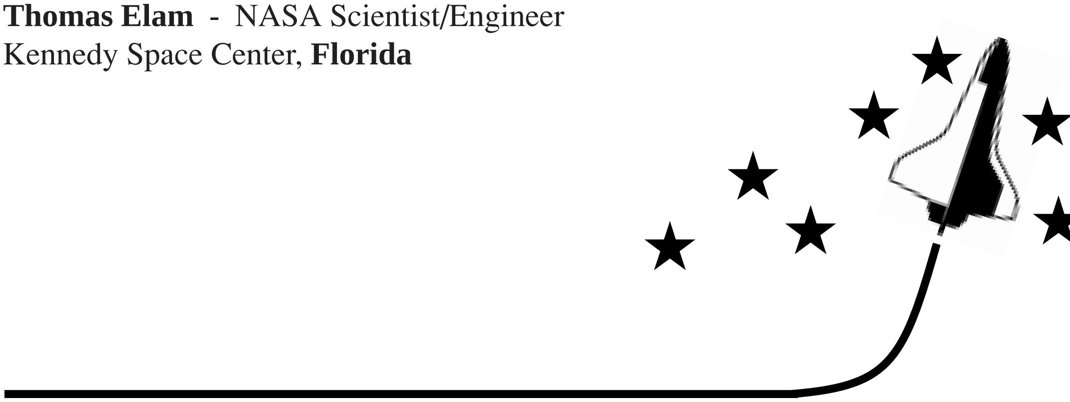

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This project was supported, in part, by the
National Science Foundation
Opinions expressed are those of the authors
and not necessarily those of the Foundation

SPINOFF 11A

Newton's Law of Cooling and Heating: Vaporization of Liquid Helium

Problem

The Kennedy Space Center (KSC) receives helium in liquid form in over-the-road tankers. As needed, this helium is vaporized, compressed, and fed into A high pressure pipeline that distributes gaseous helium to Shuttle Launch Pads A and B the three Orbital Processing Facilities (OPFs), the Vehicle Assembly Building (VAB), and other KSC support Facilities. The helium is stored at – 271.5º C. Assume that the helium tank was removed from its refrigeration unit on a day when the temperature of the surrounding air was 80º F. Also, assume that ten minutes after the helium was removed from the refrigeration unit, you measured its temperature at – 271º C. How long will it take for the temperature of the helium to reach – 269.5º C, the temperature at which helium vaporizes?

In order to solve this problem, we need to know how the temperature of a liquid changes with time. Isaac Newton first described this process in a law which later became known as Newton's Law of Cooling and Heating. If you are not familiar with Newton's Law of Cooling and Heating, click here.

Mathematical Aside

Newton's Law of Cooling and Heating

Cooling Coffee: Coffee was heated to 170º Fahrenheit in a room with air temperature of 70º F (also called ambient temperature). The temperature of the coffee was measured at two second intervals over a 50 second time period. The data collected is summarized in the table below.

Time (seconds)	Temperature (ºF)	Time (seconds)	Temperature (ºF)
0	170	26	103
2	163	28	101
4	157	30	99
6	148	32	97
8	140	34	95
10	133	36	93
12	127	38	91
14	122	40	90
16	118	42	89
18	115	44	88
20	112	46	87
22	108	48	86
24	106	50	85

Notice that the rate of cooling is faster when the difference between the temperature of the coffee and the room temperature is larger. In fact, experiment shows that the rate of change of temperature varies directly with the difference between the temperature of the coffee T and the temperature of the surrounding environment T_S. As a result, the model that relates the rate of change of temperature (T) to the time (t) is the following equation:

$$\frac{dT}{dt} = -k(T - T_S) \text{ for some constant } k$$

In this case, the ambient temperature, T_S, is the constant 70º F. To see that this is the standard differential equation with an exponential function as solution, we let $y = T - T_S$. We then have:

$$\frac{dT}{dt} = \frac{d(y + T_S)}{dt} = \frac{dy}{dt} + \frac{dT_S}{dt} = \frac{dy}{dt}, \text{ since the derivative of } T_S \text{ is 0.}$$

Therefore, if we replace $T - T_S$ by y, the differential equation becomes:

$$\frac{dy}{dt} = -ky. \qquad (1)$$

We rewrite this differential equation as $\frac{1}{y}dy = -kdt$. Integrating with respect to t, we find that the solution of the differential equation is $\ln|y| = -kt + C$. Changing this to exponential form gives $|y| = e^{-kt+C} = e^{-kt} \cdot e^{C}$. (Recall that $x^a \cdot x^b = x^{a+b}$.) Thus, $y = \pm e^C e^{-kt}$. Since C represents any real constant, $\pm e^C$ represents any non-zero real constant. Thus, $y = Ae^{-kt}$ is a solution of the differential equation (1) where A is any non-zero constant. In fact, A can even be allowed to equal 0, since the function $y = 0$ is a solution of the differential equation (1).

At time t = 0, y is the difference between the initial temperature T_0 of the coffee and the room temperature T_S. Thus, at time t = 0, y = $T_0 - T_S = Ae^{-k(0)} = A$.

Since $y = T - T_S$, Newton's Law of Cooling becomes

$$T - T_S = (T_0 - T_S)e^{-kt} \text{ or } T = T_S + (T_0 - T_S)e^{-kt}$$

Recall that the coffee was heated to 170º Fahrenheit in a room with an ambient temperature of 70º F. This means that $T_0 = 170$º F and $T_S = 70$º F. Thus, the temperature vs. time function becomes:

$$T = 70 + (170 - 70)e^{-kt} \text{ or } T = 70 + 100e^{-kt}$$

The function is complete except for the value of k. We will now estimate k by plotting the data from the table and then using trial and error to find the value of k that appears to make the model fit the data best. It will be easiest to use technology to plot the data and to graph cooling functions for trial values of k. The keystrokes to do this are given below.

Enter the data in a TI-83™ graphing calculator using L1 for the time values and L2 for the temperature values. The key strokes to do this are shown below.

STAT 4[ClrList] **2nd** L1, L2 **ENTER** {This clears the two lists.}
STAT **ENTER** {You are now ready to enter the data.} 0 **ENTER** 2 **ENTER** {Continue in this way until all times are entered in L1.} → {Now enter the temperature data in L2.}
We can now create a scatterplot of the data.

2nd STAT PLOT **ENTER** **ENTER** {Turn on the first statistics plot.}
↓ **ENTER** {Turn on the scatterplot.} ↓ **ENTER** {Let L1 be the x-values.}
→ → **ENTER**
{Now change the Window parameters to [0, 50, 5, 0, 200, 10].} **GRAPH**

Now access **Y=** and type in the function $T = 70 + 100e^{-kt}$ using a reasonable value of k. Graph the function on the screen with the scatterplot and observe how the curve fits the data. Continue this process using different values of k until you find a value for which the function appears to fit the data best.

The following function appears to fit the data quite well.

$$T = 70 + 100e^{-0.045t}$$

Now that you have gained some familiarity with Newton's Law of Cooling and Hearting we turn to another problem.

Warming orange juice: A cup of orange at 34º F is removed from a refrigerator and placed in a room with temperature of 72º F. Twenty minutes later the temperature of the orange juice is 40º F. Find the function that relates temperature *T* and time *t*.

Solution

It should be noted that Newton's Law of Cooling and Heating also applies to a quantity that is becoming warmer. Thus, the applicable model is $T = T_S + (T_0 - T_S)e^{-kt}$. At time *t* = 0, the temperature is 34º F. Thus, T_0 = 34º F and T_S = 72º F. Substituting these values into the equation gives:

$$T = 72 + (34 - 72)e^{-kt} \text{ or } T = 72 - 38e^{-kt}$$

Since *T* = 40º F when t = 20 minutes, it follows that $40 = 72 - 38e^{-k20}$. Solving for k we get:

$$k = \frac{\ln\left(\frac{32}{38}\right)}{-20} = \frac{-0.17185}{-20} = 0.0086$$

This implies that $T = 72 - 38e^{-0.0086t}$.

Click here when you are ready to return to Newton's Law of Cooling and Heating: Vaporization of Liquid Helium.